Arboricity and spanning-tree packing in random graphs with an application to load balancing

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joint work with

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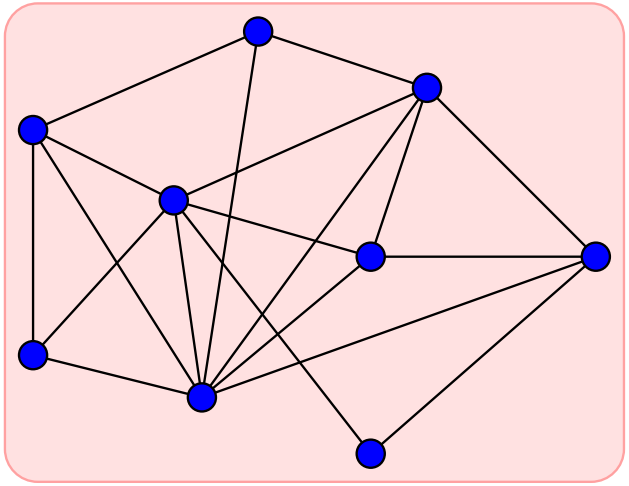
Iowa State University, February 2014

Motivation

Spanning-tree packing (STP) number

Definition

$T(G)$ = maximum number of edge-disjoint spanning trees in G.



Example

$T(G) = 2$.

Trivial upper bound

$$T(G) \leq \min\left\{\delta, \left\lfloor \frac{\bar{d}}{2} \right\rfloor\right\}, \qquad \text{where} \qquad \frac{\bar{d}}{2} = \frac{m}{n-1}.$$

An application

Measure of network strength/vulnerability.

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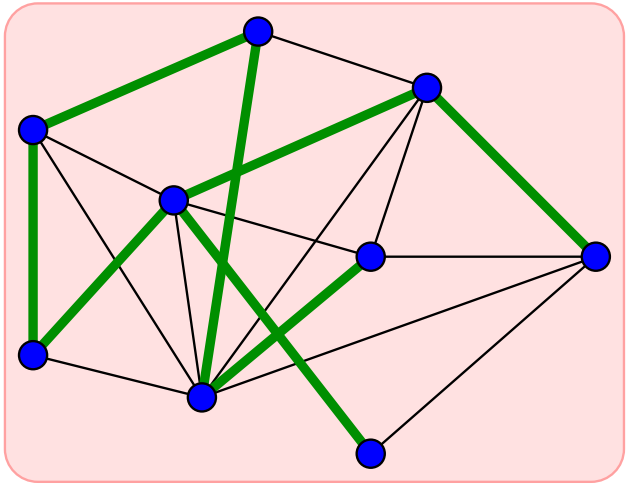

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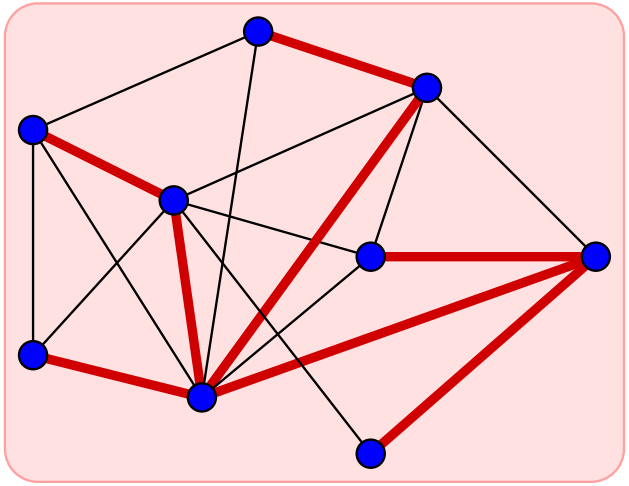

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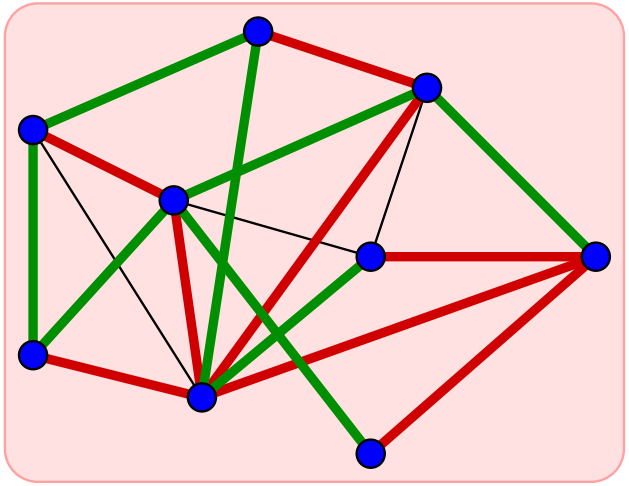

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Definition

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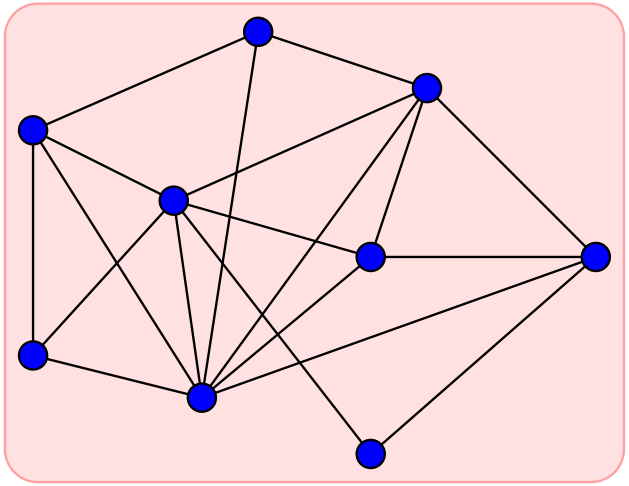
= minimum number of forests decomposing $E(G)$.



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$A(G) = 3$.

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$A(G) \geq \lceil \frac{\bar{d}}{2} \rceil$, where $\frac{\bar{d}}{2} = \frac{m}{n-1}$.

Some applications

Measure of density of subgraphs; k-orientability; load balancing.

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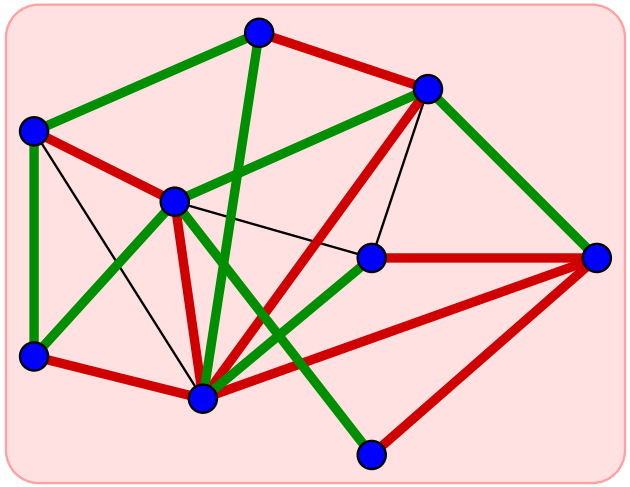

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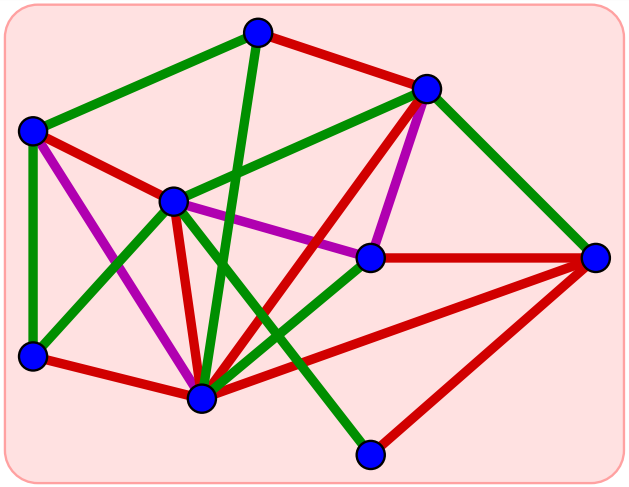

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k-orientability and load balancing

arboricity ⟷ "density" of densest subgraph

Nash-Williams'64

Hakimi'65

load balancing ⟷ k-orientability

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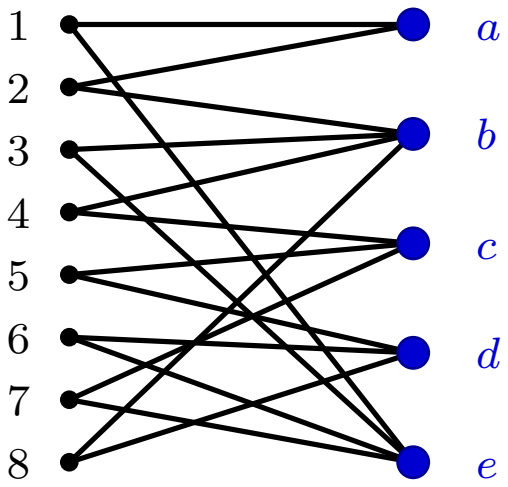

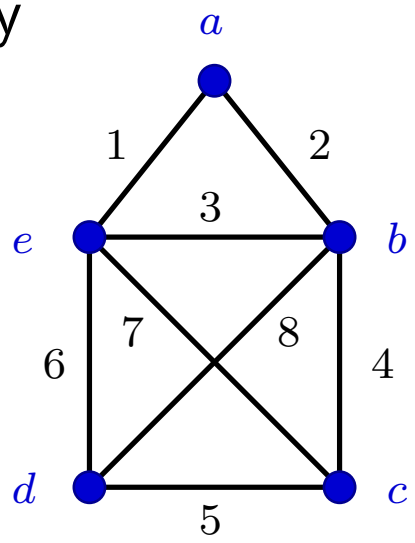

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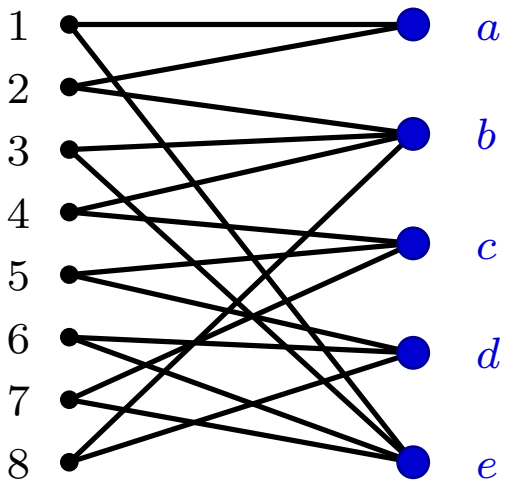

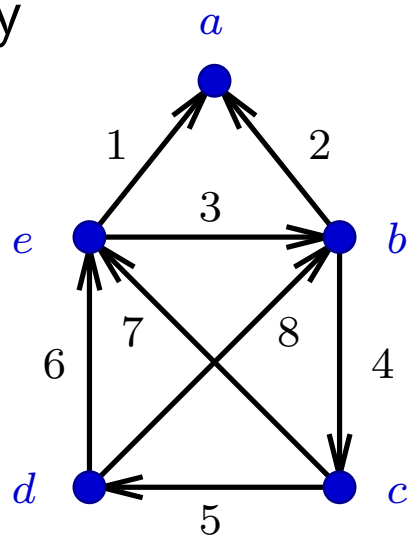

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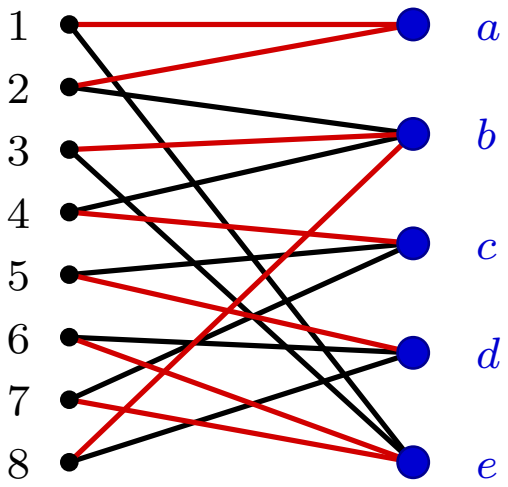

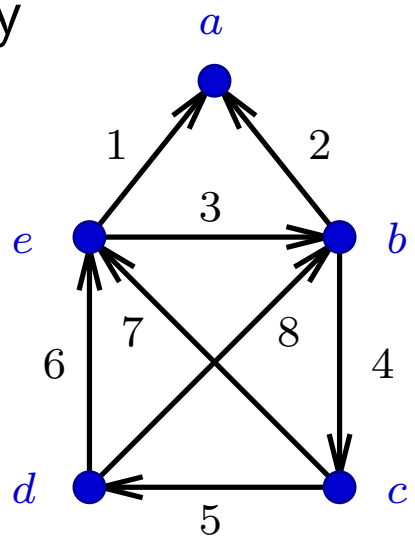

Problem and previous results

Theorem (Palmer, Spencer '95)

$T\big(\mathscr{G}(n,p)\big) = \delta$ a.a.s., for δ constant ($p \sim \log n/n$).

Theorem (Catlin, Chen, Palmer '93)

$$\begin{cases} T\big(\mathscr{G}(n,p)\big) = \lfloor \frac{\bar{d}}{2} \rfloor \\ A\big(\mathscr{G}(n,p)\big) = \lceil \frac{\bar{d}}{2} \rceil \end{cases} \quad \text{a.a.s.,} \quad \text{for } p = C(\log n/n)^{1/3}.$$

Theorem (Chen, Li, Lian '13+)

$T\big(\mathscr{G}(n,p)\big) = \delta$ a.a.s., for $p \leq 1.1 \log n/n$;
$T\big(\mathscr{G}(n,p)\big) < \delta$ a.a.s., for $p \geq 51 \log n/n$.

Question (Chen, Li, Lian)

What's the smallest p such that $T\big(\mathscr{G}(n,p)\big) < \delta$?

Our results (i)

Theorem

For every $p = p(n) \in [0,1]$, a.a.s. $T\big(\mathscr{G}(n,p)\big) = \min\left\{\delta, \left\lfloor \frac{\bar{d}}{2} \right\rfloor\right\}$.
(Same holds throughout the random graph process.)

Theorem

Let $\beta = 2/\log(e/2) \approx 6.51778$.

- If $p = \frac{\beta\left(\log n - \frac{\log\log n}{2}\right) - \omega(1)}{n-1}$, then a.a.s. $\delta \le \left\lfloor \frac{\bar{d}}{2} \right\rfloor$ in $\mathscr{G}(n,p)$.
- If $p = \frac{\beta\left(\log n - \frac{\log\log n}{2}\right) + \omega(1)}{n-1}$, then a.a.s. $\delta > \left\lfloor \frac{\bar{d}}{2} \right\rfloor$ in $\mathscr{G}(n,p)$.

(Same holds throughout the random graph process.)

Threshold for $T\big(\mathscr{G}(n,p)\big) = \begin{cases} \delta \\ \left\lfloor \frac{\bar{d}}{2} \right\rfloor \end{cases}$ at $p \sim \beta \frac{\log n}{n}$.

Our results (ii)

Theorem

- If $p = O(1/n)$, then a.a.s. $A(\mathscr{G}(n,p)) \in \{k, k+1\}$,

  where $k > \frac{\bar{d}}{2}$.

- If $p = \omega(1/n)$, then a.a.s. $A(\mathscr{G}(n,p)) \in \{\lceil \frac{\bar{d}}{2} \rceil, \lceil \frac{\bar{d}}{2} \rfloor + 1\}$.

  For most values of $p = \omega(1/n)$, a.a.s. $A(\mathscr{G}(n,p)) = \lceil \frac{\bar{d}}{2} \rceil$.

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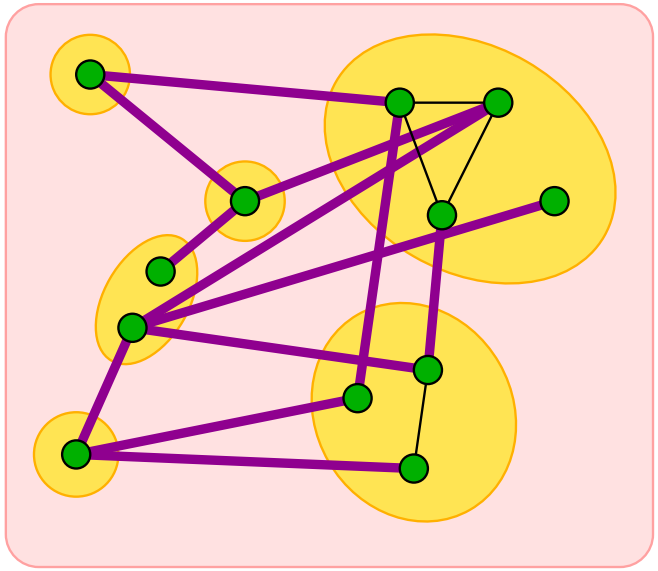
Corollary

Threshold for k-orientability of $\mathscr{G}(n,m)$ for $k \to \infty$ at $m \sim kn$.

Tutte and Nash-Williams



$\mathcal{P}$ partition of $V(G)$;
$m(\mathcal{P})$ = number of edges in $E(G)$ with ends in distinct parts of $\mathcal{P}$.

Theorem (Tutte '61; Nash-Williams '61)

A graph G has t edge-disjoint spanning trees iff every partition $\mathcal{P}$ of $V(G)$ satisfies $m(\mathcal{P}) \geq t(|\mathcal{P}| - 1)$.

$T(G)$ is given by the smallest ratio $\left\lfloor \frac{m(\mathcal{P})}{|\mathcal{P}|-1} \right\rfloor$.

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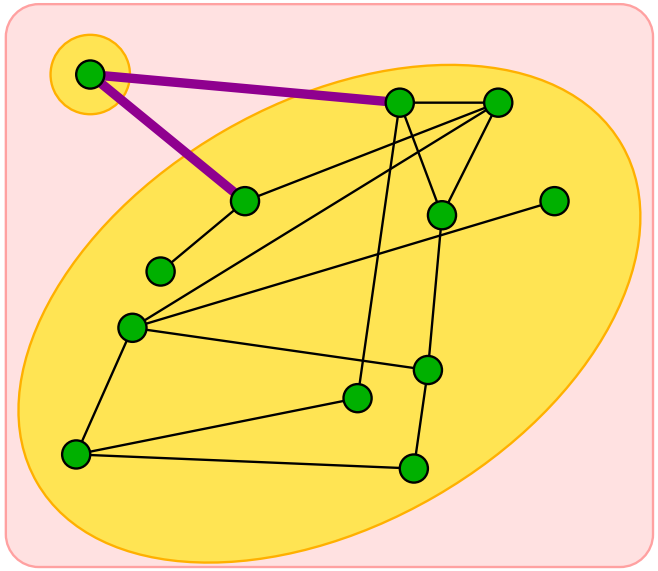

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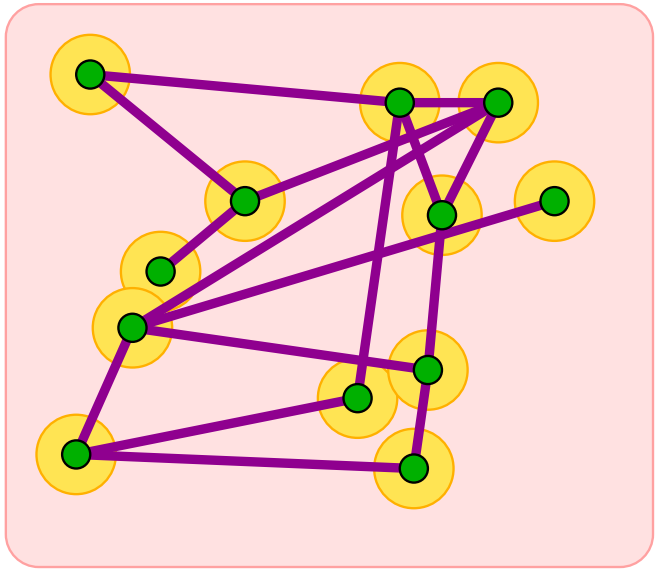

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Proof STP number $\mathscr{G}(n,p)$

Trivial range: $p \leq 0.9\frac{\log n}{n}$

$\delta = T = 0$.

Interesting range: $p \geq 0.9\frac{\log n}{n}$

Consider set of properties $\mathcal{A}$ (degrees, expansion, density...).

Prove that

- $\mathscr{G}(n,p)$ satisfies $\mathcal{A}$, a.a.s.
- $\mathcal{A}$ implies that $\forall \mathcal{P}$, the ratio $\dfrac{m(\mathcal{P})}{|\mathcal{P}|-1} \geq \begin{cases} \delta, \\ \bar{d}/2. \end{cases}$

Then, a.a.s. $T(\mathscr{G}(n,p)) = \min\{\delta, \bar{d}/2\}$.

Note: When $\delta > \bar{d}/2$, we have a full decomposition of edges.

Proof Arboricity $\mathscr{G}(n,p)$ — Range $p = \omega(1/n)$

Range $p = \omega(1/n)$

Create G' from $G = \mathscr{G}(n,p)$:

Add $o(n)$ new edges to $\mathscr{G}(n,p)$ so that

- $\delta' > \bar{d}'/2$;
- properties $\mathcal{A}$ are satisfied;
- $n-1 \mid m'$.

Then $T(G') = \bar{d}'/2 = \dfrac{m'}{n-1} = A(G')$ and

$$A(\mathscr{G}(n,p)) \in \left\{ \lceil \tfrac{m}{n-1} \rceil, \lceil \tfrac{m}{n-1} \rceil + 1 \right\}.$$

Proof Arboricity $\mathscr{G}(n,p)$ — Range $p = O(1/n)$

Theorem (Nash-Williams '64)

Edges of G can be covered by t forests iff for every non-empty $S \subseteq V(G)$ we have $|E[S]| \leq t(|S|-1)$.

$$A(G) = \max_{\emptyset \neq S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|-1} \right\rceil, \qquad \text{(where } 0/0 := 0\text{)}.$$

k-core

Largest subgraph with minimum degree k.

Theorem (follows from Hakimi '65 + Cain, Sanders, Wormald '07)

For $k \geq 2$,
if the density of the $(k+1)$-core of $\mathscr{G}(n,p)$ is at most $k + o(1)$,
then a.a.s.
$\mathscr{G}(n,p)$ has no subgraph with density more than $k + o(1)$.

Proof Arboricity $\mathscr{G}(n,p)$ — Range $p = O(1/n)$

Summary of the argument

If $p \leq 1/n$, then $A \in \{1,2\}$ (easy).

If $p > 1/n$:

- Find k such that
  - the $(k+1)$-core of $\mathscr{G}(n,p)$ has density at most $k + o(1)$;
  - the k-core of $\mathscr{G}(n,p)$ has density greater than $(k-1)$.
- Then the densest subgraph of $\mathscr{G}(n,p)$ has density $> (k-1)$ and $\leq k + o(1)$.
- So $A \in \{k, k+1\}$.

Further work

- Extend results to random hypergraphs (easy).
- Extend results to other families of random graphs
  (work in progress for random geometric graphs, sparse graphs with a fix degree sequence).
- Study other graph parameters with similar characterisations following from matroid union
  (work in progress for random directed graphs).

Thank you